THE GAME OF "GOSSIP" ANALYZED BY THE THEORY
OF INFORMATION

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ABSTRACT

This paper deals with an analysis of a distributed
model of the game of "gossip", in which a message is passed
through a line of indivi uals, and tne final (in general,
garbled) result is compared witn tne original ungarbled
messa,e. The deterioration of information (def ned in the
sense of Shannon and Wiener) along the line is calculatod,
and exact as well as asymptotic formulas suggest approximate
linear electric network analogues.

I. Object of this Paper

In discussing communication nets involving human
beings and/or other automata one is te rted to call upon
analogies from the theory of linear electric network to
describe the mutual exchange of information on terms of
analogous current or voltage propagation. In this paper
we consider a particular type of communication system, and
investirate the approximations under whicn for that particular
case an analogy can rigorously be drawn, using tne strict definition
of informat n proposed by Shannon and Wiener [1]

II. The Theory of Information

Consider a communication system (Fig. 1) described
by the square nxn matrix

$$P(i,j) = \text{joint probability that the } i^{th} \text{ symbol is sent, and,} \qquad (1)$$
$$\text{as a result of this, the } j^{th} \text{ symbol received.}$$

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(1) Confer list of references at end of paper.

Define

$$P(i) = \sum_{j=1}^{n} P(i,j) = \text{a-priori probability of sending symbol } i. \qquad (2)$$

$$Q(j) = \sum_{i=1}^{n} P(i,j) = \text{a-priori probability of receiving symbol } j. \qquad (3)$$

$$Q_i(j) = P(i,j)/P(i) = \text{conditional probability that } j \text{ is} \qquad (4)$$

recieved if i is sent.
It is evidently necessary that

$$Q_i(j) = 1, \quad (i=1,2,\ldots,n). \qquad (5)$$

Shannon has shown [2] that the information rate

of the communication system of Fig. 1 should be defined as

$$R = H - E, \text{ where} \qquad (6)$$

$$H = -\sum_{j=1}^{n} Q(j) \log Q(j) \text{ is the negentropy of the received symbols}[3] \qquad (7)$$

$$E = -\sum_{i,j=1}^{n} P(i,j) \log Q_i(j) \text{ is the equivocation of the system.} \qquad (8)$$

In the special case when all the symbols to be sent

are equally likely we have

$$P(i) = 1/n. \qquad (9)$$

Therefore, $P(i,j) = (1/n) Q_i(j)$, giving

$$E = -(1/n) \sum_{i,j,=1}^{n} Q_i(j) \log Q_i(j). \qquad (10)$$

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(2) Cf. ref. 7, pg. 409; also see ref. 5 for a discussion of

the arbitrariness of the choice of the definition of information.

Shannon uses $H(y)$ for our H, and $H_x(y)$ for our E. Shannon

uses logs to the base 2; we employ natural logs.

(3) The term "negentropy" is due to Brillouin, ref. 2.

Also, if in addition, $v_i(j) = v_j(i)$

$$Q(j) = \sum_{i=1}^{n} P(i) Q_i(j) = (1/n) \sum_{j=1}^{n} Q_i(j) = (1/n) \text{ by (5).} \qquad (11)$$

In this particular case, therefore, H=logn, so that

$$R = \log n + (1/n) \sum_{i,j=1}^{n} Q_i(j) \log Q_i(j). \qquad (12)$$

III. Model for the Game of Gossip

The game is post lated to co sist in selectin a symbol (message) at random from a set of n symbols, such that all n symbols are a-priori equally likely, a d propagating it down the line of individuals. The participa ts are assumed to know what the a-priori possible messages are, a d to realize that they are a-priori equally likely.

Let x measure distance from the beginning, or origin of the line.

We now make the idealizing postulate that if the symbol arriving at x is the ith symbol, the the probability that an arbitrary symbol j arrives at x+dx is given by the conditional probability matrix

$$Q_i(j)_{x,x+dx} = \begin{bmatrix} 1-\alpha(n-1)dx & \alpha dx & \cdots & \alpha dx \\ \alpha dx & 1-\alpha(n-1)dx & \cdots & \alpha dx \\ \alpha dx & \alpha dx & & 1-\alpha(n-1)dx \end{bmatrix}$$

Here α should be tho ght of as a number proportional to the de sity of human beings per unit distance a o g the li e. In the realistic case distance would be measured as proportional to the number of human beings. Therefore α plays the role of a characteri tic frailty (independent of n) which must be found by experime t [4].

Symbolically

$$Q_i(j)_{x,x+dx} = (1-\alpha n dx)\, \delta_{ij} + \alpha dx \quad (5).$$ (13)

Let

$$f_{ij}(x) = \text{conditional probability that } j \text{ will be received at } x$$ (14)

if i is inserted at $x=0$.

Then, using (13), (5)

$$f_{ij}(x+dx) = \sum_{k=1}^{n} f_{ik}(x)\, Q_k(j)_{x,x+dx} = f_{ij}(x) - f_{ij}(x)\, \alpha n dx + \alpha dx$$ (15)

Thus $f_{ij}(x)$ satisfies the differential equation

$$f'_{ij}(x) + \alpha n f_{ij}(x) - \alpha = 0$$ (16)

The solution, with the initial condition $f_{ij}(0) = \delta_{ij}$ is

$$f_{ij}(x) = e^{-\alpha n x}\delta_{ij} + (1-e^{-\alpha n x})/n.$$ (17)

IV. Information Flow in the Game of Gossip

According to (10), if $E(x)$ is the equivocation in the stretch of line $(0.x)$,

$$E(x) = -(1/n)\sum_{i,j=1}^{n} f_{ij}(x)\log f_{ij}(x).$$ (18)

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(4) Although an exact solution for the case of discrete individuals could have been obtained the continous case was purposely chosen because it is simpler.

(5) δ_{ij} is the Kronecker delta: $\delta_{ij} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i\neq j \end{cases}$.

Substituti: (17), manipulating algebraically, and inserting

into (12) gives as the basic result

$$R_n(x) = \frac{1 + (n-1)e^{-anx}}{n} \log \left[1 + (n-1)e^{-anx}\right] +$$ (19)

$$+ \frac{(n-1)(1 - e^{-anx})}{n} \log (1 - e^{-anx}).$$

Here $R_n(x)$ is the information available at x regarding the

original message. A plot of the function is given in Fig. 2.

As obvious checks we have

$$\lim_{x \to 0} R_n(x) = \log n$$ (20)

$$\lim_{x \to \infty} R_n(x) = 0$$ (21)

$$R_1(x) = 0$$ (22)

$$\lim_{n \to \infty} R_n(x) = 0 \quad \text{when } x \neq 0.$$ (23)

It is interesting to evaluate the derivative

of $R_n(x)$. This is

$$R_n'(x) = \alpha(n-1) e^{-anx} \log \left[\frac{1 - e^{-anx}}{1 + (n-1) e^{-anx}}\right]$$ (24)

This shows that

(a) $R_n(x)$ is strictly decreasing as a function of x

except when $\alpha(n-1) = 0$ in which case $R_n(x) = $ const.

(b) The slope of $R_n(x)$ at the origin is infinite when $\alpha(n-1) \neq 0$.

V. Estimates and Asymptotic Relations

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The following two relations are easily obtained

from (19):

For small x:

$$R_n(x) = \log n + a\,(n-1)\,x\log x + 0\,(x^2\log x). \tag{25}$$

For large n, and ax larger than some positive quantity

independent of n:

$$R_n(x) = (3n/2)e^{-2anx} + 0\,(e^{-2anx}). \tag{26}$$

An interesting relation that can also be

obtained is

$$\lim_{n\to\infty} \frac{R_n\left[\dfrac{1}{a(n-1)}\right]}{R_n(0)} = 1/e \doteq 0.37. \tag{27}$$

This indicates that for large n if we go out a distance

$1/[a(n-1)]$ from the origin we will find that the information

has deteriorated to approximately 37% of its initial value.

VI. Linear Electric Network Analogies

Linear electric net-work analogy approximations

can be obtained by exploiting either (26) or (27). (25) cannot

be used because the $x\log x$ term prevents it from being the

solution of a linear differential equation with constant

coefficients.

An electric analogy could use either time or

distance as the quantity analogous to x. If distance is

used, we have the case of an electric transmission line

with attenuation. Recognizing that a series combination

of resistance R and capacitance C has a time constant RC
seconds (time in which the original capacitor voltage goes
to $1/e \doteq 0.37$ of its initial value) a simple lumped analogy
making use of (27) can be obtained. Consider namely a
capacitance of value 1 farad and leakage resistance
$[1/\alpha(n-1)]$ ohms (Fig. 3). If the capacitance is originally
charged to $\log_e n$ volts then the capacitor voltage as a function
of time is approximately analogous to the information in the
gossip line as a function of distance.

Vii. Supplementary Remark

Of the several possible modifications of the
preceding analysis that come to mind the most obvious
one is the adoption of a different $q_i(j)$ $x, x+dx$.
This will simply change the differential equation (16),
but the rest of the work would be the same. One possible
example is to make α a function of x. This would reflect
the effect of relative position in the group on the frailty
of the individual.

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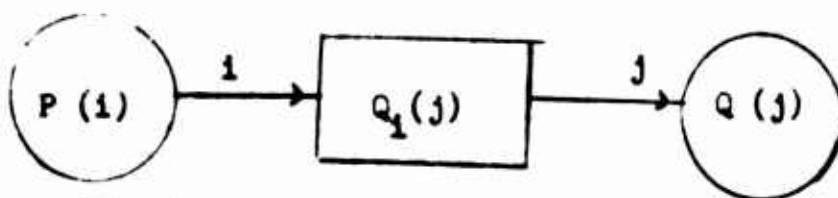12.   N. Wiener, "Cybernetics", John Wiley, New York, 1948, pp. 74-81.
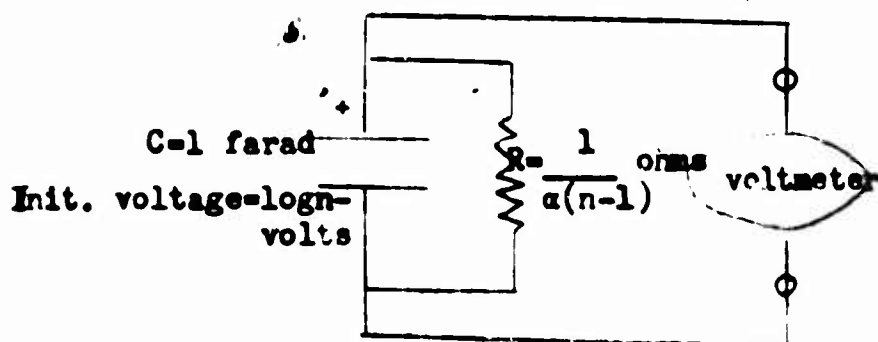
Fig. 1.   Fundamental Communication system

Fig. 2   Approximate anology for gossip line having
frailty per unit distance α, for case of n possible
messages. Here time is the quantity analogous to
distance along the gossip line, and voltage is
analogous to information.

Fig. 3   Information along gossip line as a function of distance
from origin.